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# Graph $C^*$-Algebras with Real Rank Zero

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Given a row-finite directed graph $E$, a universal $C^*$-algebra $C^*(E)$ generated by a family of partial isometries and projections subject to the relations determined by $E$ is associated to the graph $E$. The Cuntz–Krieger algebras are those graph $C^*$-algebras associated to some finite graphs. We prove that a graph $C^*$-algebra $C^*(E)$ has real rank zero in the sense that the set of invertible self-adjoint elements is dense in the set of all self-adjoint elements in $C^*(E)$ (or in the unitization of $C^*(E)$ if $C^*(E)$ is nonunital) if and only if $E$ satisfies a loop condition (K) that is analogous to the condition for a finite $\{0, 1\}$ matrix $A$ under which Cuntz analyzed the ideal structure of the Cuntz–Krieger algebra $\mathcal{O}_A$. 

*Key Words:* real rank; directed graph; graph $C^*$-algebra.

## 1. INTRODUCTION

For a $\{0, 1\}$-matrix $A$ satisfying a certain condition (I), Cuntz and Krieger [3] construct the Cuntz–Krieger algebra $\mathcal{O}_A$ and Cuntz [2] determines the ideal structure of $\mathcal{O}_A$ for a matrix $A$ satisfying condition (II) which is stronger than (I). Graph $C^*$-algebras [13] for locally finite directed graphs with no sinks are introduced as an extended version of the Cuntz–Krieger algebras. In [12] Kumjian *et al.* define the graph

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$C^*$-algebra $C^*(E)$ for any countable row-finite directed graph $E$ possibly with sinks and prove that if $E$ satisfies a certain loop condition (L), which is analogous to the condition (I), then an analogue of the Cuntz–Krieger uniqueness theorem holds for $C^*(E)$. A graph $C^*$-algebra $C^*(E)$ is generated by a Cuntz–Krieger $E$-family of projections and partial isometries subject to the relations determined by $E$ or by the edge matrix of $E$ if $E$ has no sinks. Moreover $C^*(E)$ is a universal $C^*$-algebra for Cuntz–Krieger $E$-family. In [13], a groupoid $\mathcal{G}$ associated to a locally finite directed graph $E$ with no sinks is considered and it is shown in the same paper that the groupoid $C^*$-algebra $C^*(\mathcal{G})$ is a universal $C^*$-algebra generated by a Cuntz–Krieger $A$-family of projections and partial isometries, where $A$ is the edge matrix for the graph $E$. It then follows that the two $C^*$-algebras $C^*(\mathcal{G})$ and $C^*(E)$ are isomorphic, and hence some knowledge for groupoid $C^*$-algebras could be applied to analyze the ideal structure of graph $C^*$-algebras when $E$ satisfies condition (K), which is analogous to the condition (II) of a matrix $A$ generating a Cuntz–Krieger algebra $\mathcal{O}_A$.

Since graph $C^*$-algebras $C^*(E)$ have partial isometries and projections as their generators one might expect that most of the graph $C^*$-algebras would have enough projections and they might have real rank zero. In fact, the Cuntz algebras $\mathcal{O}_n$ are known to have real rank zero since they are purely infinite and simple [15], but the Toeplitz algebra and its quotient algebra $C(\mathbb{T})$ are graph $C^*$-algebras with real rank greater than 0. Thus the question in view of abundance of projections is to find a condition in terms of the graph under which the associated graph $C^*$-algebra has real rank 0. It is proved in [7] that if $C^*(E)$ has real rank zero then $E$ should satisfy condition (K), and the converse is true if $C^*(E)$ has finitely many ideals. In this paper we show that for a locally finite directed graph $E$ the graph $C^*$-algebra $C^*(E)$ has real rank zero exactly when $E$ satisfies condition (K). Given a directed graph $E$, it is immediate to see whether $E$ satisfies condition (K). Hence our theorem provides a very convenient way to know that the real rank of a graph $C^*$-algebra will be zero or not.

It is known [8] that a separable nuclear simple $C^*$-algebra $A$ is purely infinite if and only if $A \simeq A \otimes \mathcal{O}_\infty$. It then follows from our theorem and a result in [6] that if a graph $C^*$-algebra $C^*(E)$ is purely infinite in the sense of [9] (simple or nonsimple) its real rank is zero. Hence we have that $C^*(E)$ is purely infinite if and only if $C^*(E) \simeq C^*(E) \otimes \mathcal{O}_\infty$ by [10, Corollary 9.4].

## 2. PRELIMINARIES

A directed graph consists of vertices and directed edges between them. More formally, a directed *graph* $E$ is a quadruple $(E^0, E^1, r, s)$, where $E^0$

and $E^1$ are countable sets of *vertices* and *edges*, and $r, s: E^1 \to E^0$ are the *range* and *source* maps, respectively.

A vertex $v$ that emits no edges is called a *sink*. A graph $E$ is *row-finite* if every vertex emits only finitely many edges. A row-finite graph $E$ is *locally finite* if every vertex receives only finitely many edges and *finite* if $E^0$ is a finite set.

We call a sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ of edges $\alpha_i$, where $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leqslant i \leqslant n-1$, a *path* of length $|\alpha| = n$. A finite path $\alpha$ with $|\alpha| > 0$ is called a *loop* based at $v$ if $s(\alpha) = r(\alpha) = v$. We denote $\alpha^0$ the set of vertices on the loop $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, that is, $\alpha^0 = \{v \in E^0 \mid v = s(\alpha_i),\ 1 \leqslant i \leqslant n\}$ as in [6]. A loop $\alpha$ is *simple* if the vertices $s(\alpha_i)$, $1 \leqslant i \leqslant |\alpha|$, are distinct. For a subgraph $F$ of $E$, an *exit* of $F$ is an edge $e$ of $E$ with $s(e) \in F^0$ and $e \notin F^1$.

Let $E^n$ be the set of paths of length $n$. By convention $E^0$ is identified with the set of paths of length 0. The range map $r$ and the source map $s$ extend to $E^* = \bigcup_{n \geqslant 0} E^n$ by $r(\alpha) = r(\alpha_{|\alpha|})$ and $s(\alpha) = s(\alpha_1)$.

A graph $E$ is said to satisfy *condition* (L) if every simple loop in $E$ has an exit, and *condition* (K) if for each vertex $v$ on a loop there exist at least two distinct loops $\alpha, \beta$ based at $v$ such that $r(\alpha) = r(\beta) = s(\alpha) = s(\beta) = v$, $r(\alpha_i) \neq v$ for $1 \leqslant i < |\alpha|$ and $r(\beta_j) \neq v$ for $1 \leqslant j < |\beta|$.

Given a row-finite graph $E$, a *Cuntz–Krieger E-family* consists of a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ indexed by vertices and a family of partial isometries $\{s_e : e \in E^1\}$ indexed by edges satisfying the *Cuntz–Krieger E-relations*:

$$s_e^* s_e = p_{r(e)} \qquad \text{for} \quad e \in E^1,$$

$$p_v = \sum_{\{e:\, s(e) = v\}} s_e s_e^* \qquad \text{for} \quad v \in E^0, \qquad \text{which is not a sink.}$$

In [12] Kumjian *et al.* showed that there exists a $C^*$-algebra $C^*(E)$ generated by a Cuntz–Krieger $E$-family $\{s_e, p_v\}$, which is universal in the sense that for any Cuntz–Krieger $E$-family $\{t_e, q_v\}$ in a $C^*$-algebra $A$, there exists a $*$-homomorphism $\pi$: $C^*(E) \to A$ such that $\pi(s_e) = t_e$ and $\pi(p_v) = q_v$. This $C^*$-algebra $C^*(E)$ is called the *graph $C^*$-algebra of E*. It is useful to note that the linear span of the set $\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}$ is dense in $C^*(E)$ and the finite sums of projections $\{p_v\}$ forms an approximate unit for $C^*(E)$. Moreover graph $C^*$-algebras are nuclear since they are stably isomorphic to the crossed products of AF algebras by the integer group $\mathbb{Z}$ (see [11]).

## 3. IDEAL STRUCTURE OF GRAPH $C^*$-ALGEBRAS

We need to review the ideal theory of a graph $C^*$-algebra $C^*(E)$ for a directed graph $E$ with no sinks [13].

For vertices $v, w$ of $E$, we write $v \geqslant_E w$ if $v = w$ or there is a path $\alpha$ with $s(\alpha) = v$ and $r(\alpha) = w$. Then a subset $H$ of $E^0$ is called *hereditary* if

$$v \geqslant_E w \quad \text{and} \quad v \in H \Rightarrow w \in H,$$

and *saturated* if

$$s^{-1}(v) \neq \varnothing \quad \text{and} \quad r(s^{-1}(v)) \subset H \Rightarrow v \in H.$$

Given set inclusion as the order relation, the set of all hereditary saturated subsets has the complete lattice structure. For a subset $N$ of $E^0$ there exists a smallest hereditary saturated subset of $E^0$ containing $N$ which we call the *hereditary saturation* of $N$ and write $\langle N \rangle$.

If $E$ is row-finite, every hereditary saturated subset $H$ of $E^0$ gives rise to a closed two-sided ideal $I(H)$ of $C^*(E)$, where

$$I(H) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}.$$

Furthermore the following is known. By an ideal we always mean a closed two-sided one.

THEOREM 3.1 ([13, THEOREM 6.6] OR [14, THEOREM 2.2]). *Let $E$ be a locally finite directed graph with no sinks. Then the map $H \mapsto I(H)$ described above is injective, and the quotient algebra $C^*(E)/I(H)$ is isomorphic to $C^*(F)$ of the directed graph $F := (E^0 \setminus H, \{e \mid r(e) \notin H\})$. The ideal $I(H)$ is strong Morita equivalent to $C^*(G)$ of the directed graph $G := (H, \{e \mid s(e) \in H\})$. Moreover, if $E$ satisfies condition* (K) *then the map $H \mapsto I(H)$ is an isomorphism of the lattice of hereditary saturated subsets of $E^0$ onto the lattice of ideals in $C^*(E)$.*

Using this theorem we will see the ideal structure of graph $C^*$-algebra s by investigating hereditary saturated subsets of vertices of the graphs. In the following we define some notions on graphs which will facilitate counting the number of hereditary subsets of vertices in a graph.

DEFINITION 3.1. Let $F$ be a subgraph of $E$. The *loop completion* $\ell_E(F)$ of $F$ in $E$ is a subgraph of $E$ that is the union of $F$ and all the loops of $E$ based at the vertices of $F$. The *loop contraction* $\ell c(F)$ of $F$ is a graph obtained by shrinking each loop in $F$ to one consisting of only one edge.

More formally, the loop contraction can be defined as follows. First we define relations $\sim_0$ and $\sim_1$ on vertex set and edge set of $F$, respectively;

$$v \sim_0 w \Leftrightarrow \ell_F(\{v\}) = \ell_F(\{w\}),$$
$$e \sim_1 f \Leftrightarrow s(e) \sim_0 s(f), r(e) \sim_0 r(f).$$

These are easily seen to be equivalence relations. We extend the range map and the source map to those on the equivalence classes naturally; $r([e]) = [r(e)]$ and $s([e]) = [s(e)]$. Then the quadruple $[F] := (F^0/\sim_0, F^1/\sim_1, r, s)$ becomes a directed graph and has, if any, only simple loops consisting of one edge. Then the loop contraction $\ell c(F)$ of $F$ is defined to be the graph $[F]$.

Note that under the quotient maps $f_0 : v \in F^0 \mapsto [v] \in F^0/\sim_0$ and $f_1 : e \in F^1 \mapsto [e] \in F^1/\sim_1$, the order relation of vertices in $F$ is preserved:

$$v \geqslant_F w \Leftrightarrow [v] \geqslant_{\ell c(F)} [w].$$

Thus $F$ and its loop contraction $\ell c(F)$ have the isomorphic lattice structure of hereditary subsets of vertices.

LEMMA 3.1. *Let $E$ be a locally finite directed graph and $F$ a subgraph of $E$.*

1. *If $F$ is a finite subgraph of $E$, then the number of hereditary subsets of $\ell_E(F)^0$ is finite.*
2. *Let $G$ be a graph obtained from $F$ by adding some edges between some pairs of vertices of $F$. Then the number of hereditary subsets of $G^0$ is less than or equal to that of $F^0$.*

*Proof.* (1) Since the cardinality of the vertex set $\ell c(\ell_E(F))^0$ is not larger than that of $F^0$ and two graphs $\ell c(\ell_E(F))$ and $\ell_E(F)$ have the isomorphic lattice structure of hereditary vertex subsets, the number of hereditary subsets of $\ell_E(F)^0$ is finite.

(2) Adding edges implies that

$$v \geqslant_F w \Rightarrow v \geqslant_G w.$$

Let $H$ be a $G$-hereditary subset, that is, a hereditary subset of $G^0$. Then

$$\begin{aligned} v \geqslant_F w, \quad v \in H &\Rightarrow v \geqslant_G w, \quad v \in H \\ &\Rightarrow w \in H. \end{aligned}$$

Therefore every $G$-hereditary subset is a $F$-hereditary subset. ∎

DEFINITION 3.2. Let $F$ be a subgraph of $E$. The *exit completion* $F_e$ of $F$ is a subgraph of $E$ obtained by adding to $F$ all the edges $e \in E^1$ such that $s(e) = s(f)$ for some $f \in F^1$ and their range vertices $r(e)$. If $F = F_e$ we call $F$ *exit complete*.

Note that $F_e$ does not necessarily contain all the exits of $F$.

PROPOSITION 3.1. *Let $F$ be an exit complete subgraph of $E$ and $B$ be the $C^*$-subalgebra of $C^*(E)$ generated by $\{s_e, p_v \mid e \in F^1, v \in F^0\}$. Then there is a $*$-isomomorphism $h$: $C^*(F) \to B$ such that*

$$h(t_e) = s_e, \qquad h(q_v) = p_v, \qquad e \in F^1, \qquad v \in F^0,$$

*where $\{t_e \mid e \in F^1\} \cup \{q_v \mid v \in F^0\}$ is the Cuntz–Krieger $F$-family generating $C^*(F)$.*

*Proof.* Let $\{s_e \mid e \in E^1\} \cup \{p_v \mid v \in E^0\}$ be the Cuntz–Krieger $E$-family generating $C^*(E)$. If $v \in F^0$ then either $v$ is a sink of $F$ or $s^{-1}(v) = \{e \in E^1 \mid s(e) = v\} \subset F^1$ since $F$ is exit complete. For each nonsink $v \in F^0$, the Cuntz–Krieger $E$-relation at $v$, $p_v = \sum_{\{e \in E^1 \mid s(e) = v\}} s_e s_e^*$, coincides with the Cuntz–Krieger $F$-relation $p_v = \sum_{\{e \in F^1 \mid s(e) = v\}} s_e s_e^*$ at $v$. Thus the elements $\{s_e \mid e \in F^1\} \cup \{p_v \mid v \in F^0\}$ in $C^*(E)$ forms a Cuntz–Krieger $F$-family. Now let $C^*(F)$ be represented faithfully in a Hilbert space $H = \bigoplus_{v \in F^0} H_v$, where $H_v = q_v H$ is an infinite-dimensional Hilbert space corresponding to each vertex $v \in F^0$ such that $H_v = \bigoplus_{\substack{s(e) = v \\ e \in E^1}} H_e$ and $t_e (e \in F^1)$ is an isometry of $H_{r(e)}$ onto $H_e$. Similarly, for each $w \in E^0 \setminus F^0$, $f \in E^1 \setminus F^1$ take infinite dimensional Hilbert spaces $H_f$ and $H_w = \bigoplus_{s(f) = w} H_f$. Put $K = \bigoplus_{v \in E^0} H_v$. Choose an isometry $S_f$ of $H_{r(f)}$ onto $H_f$ and let $P_w$ be the orthogonal projection from $K$ onto $H_w$. Then the projections $\{q_v, P_w \mid v \in F^0, w \in E^0 \setminus F^0\}$ and the partial isometries $\{t_e, S_f \mid e \in F^1, f \in E^1 \setminus F^1\}$ form a Cuntz–Krieger $E$-family in the Hilbert space $K$. Hence there exists a $*$-isomomorphism $g$ from $C^*(E)$ onto the $C^*$-algebra generated by these projections and partial isometries such that

$$h(p_v) = \begin{cases} q_v, & v \in F^0 \\ P_v, & v \in E^0 \setminus F^0 \end{cases} \qquad \text{and} \qquad h(s_e) = \begin{cases} t_e, & e \in F^1 \\ S_e, & e \in E^1 \setminus F^1. \end{cases}$$

Note here that $C^*(F) = g(B)$ since $g(s_e) = t_e$, $e \in F^1$ and $g(p_v) = q_v$, $v \in F^0$. From the universal property of the $C^*$-algebra $C^*(F)$ it follows that $B$ has the same property. Then the uniqueness of such a universal $C^*$-algebra implies that $g$ should be an isomorphism from $B$ onto $C^*(F)$ and $h = g^{-1}$ is the desired isomorphism. ∎

## 4. REAL RANK OF GRAPH $C^*$-ALGEBRAS

The *real rank* of a unital $C^*$-algebra $A$, denoted by RR($A$), is defined by Brown and Pedersen [1] to be the smallest integer $n$ such that for each $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ of self-adjoint elements in $A$ and every $\varepsilon > 0$, there is an $(n+1)$-tuple $(y_1, \ldots, y_{n+1})$ of self-adjoint elements in $A$ such that $\sum y_k^2$ is invertible and

$$\left\| \sum_{k=1}^{n+1} (x_k - y_k)^2 \right\| < \varepsilon.$$

For a nonunital $C^*$-algebra $A$, its real rank is defined to be $\mathrm{RR}(A^\sim)$, where $A^\sim$ is the unitization of $A$.

In particular unital $C^*$-algebra s $A$ of real rank zero are those in which every self-adjoint element can be arbitrarily closely approximated by invertible self-adjoint elements, or equivalently every hereditary $C^*$-subalgebra of $A$ contains an approximate unit consisting of projections [1]. Hence $RR(A) = 0$ implies that $A$ contains fairly many projections. All AF algebras and all von Neumann algebras have real rank zero. Recall that a simple $C^*$-algebra is said to be *purely infinite* if each of its nonzero hereditary $C^*$-subalgebra has an infinite projection. It is known that a purely infinite simple $C^*$-algebra is of real rank zero [15]. For example, the Cuntz algebras $\mathcal{O}_n$ $(2 \leqslant n \leqslant \infty)$ have real rank zero. Also some of the Cuntz–Krieger algebras $\mathcal{O}_A$ are known to have real rank zero (see [7, Corollary 4.7]).

Since graph $C^*$-algebras $C^*(E)$ have projections and partial isometries as generators like Cuntz–Krieger algebras one might expect that most of the graph $C^*$-algebras would contain enough projections and have real rank zero. The question we are interested in is to find a graph theoretic condition under which the graph $C^*$-algebras are of real rank zero, and the following theorem is the answer to this question.

THEOREM 4.1. *Let $E$ be a locally finite directed graph. Then the following are equivalent.*

1. *$C^*(E)$ is of real rank zero.*
2. *$E$ satisfies condition* (K).
3. *$C^*(E)$ has no quotients containing a corner that is $*$-isomorphic to $M_n(C(\mathbb{T}))$.*

A partial result has been known for some graph $C^*$-algebras.

THEOREM 4.2 ([7, THEOREMS 4.3 and 4.6]). *Let $E$ be a locally finite directed graph with no sinks. If* $\mathrm{RR}(C^*(E)) = 0$ *then $E$ satisfies condition* (K). *Conversely if $E$ satisfies condition* (K) *and $C^*(E)$ has only finitely many ideals then* $\mathrm{RR}(C^*(E)) = 0$. *In particular, if $E$ is a finite graph then* $\mathrm{RR}(C^*(E)) = 0$.

Recall that if $E$ is a locally finite directed graph with no sinks and $H$ is a hereditary saturated subset of $E^0$ then the quotient algebra $C^*(E)/I(H)$ is isomorphic to $C^*(F)$ where $F := (E^0 \setminus H, \{e \mid r(e) \notin H\})$. We simply call $F$ a *quotient graph* of $E$.

PROPOSITION 4.1. *Let $E$ be a locally finite graph with no sinks and $H$ be a hereditary saturated vertex subset of $E^0$. If both the ideal $I(H)$ and the quotient algebra $C^*(E)/I(H) \cong C^*(F)$ have real rank zero, then* $\mathrm{RR}(C^*(E)) = 0$.

*Proof.* By [1, Theorem 3.14], it is enough to show that every projection in the quotient algebra $C^*(E)/I(H) \cong C^*(F)$ lifts to a projection in $C^*(E)$.

For a row-finite graph $E$ with no sinks, the $K_0$ group $K_0(C^*(E))$ is generated, as an Abelian group, by the equivalence classes $\{[p_v]: v \in E^0\}$ subject to the relation

$$[p_v] = \sum_{e:\, s(e) = v} [p_{r(e)}]$$

by [14, Theorem 3.1]. It then follows that the induced homomorphism from $K_0(C^*(E))$ to $K_0(C^*(E)/I(H))$ is surjective. By [1, Proposition 3.15] we see that every projection in $C^*(E)/I(H)$ lifts to a projection in $C^*(E)$. ▮

*Proof of Theorem* 4.1. Let $H$ be the hereditary saturation of the sinks of $E$. Then the ideal $I(H) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$ is AF by [12, Corollary 2.2] and the quotient algebra $C^*(E)/I(H)$ is isomorphic to $C^*(G)$, where $G = (E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\})$ is a quotient graph of $E$ with no sinks [7, Theorem 3.5]. Since $RR(I(H)) = 0$ and every projection in $C^*(G)$ lifts to a projection in $C^*(E)$ by [1, Corollary 3.16] because $K_1(I(H)) = 0$, it follows from [1, Theorem 3.14] that $RR(C^*(E)) = 0$ if and only if $RR(C^*(G)) = 0$. Also, it is easy to see that $E$ satisfies condition (K) exactly when $G$ does so. Thus we may assume that $E$ has no sinks for the proof of (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2): This follows from Theorem 4.2.

(2) $\Rightarrow$ (1): Assume that $E$ satisfies condition (K). Let $s$ be a self-adjoint element of $C^*(E)^\sim$. Since the linear span of the set $\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}$ is dense in $C^*(E)$, there is a finite sum $s' = \sum \lambda_{\alpha\beta} s_\alpha s_\beta^* + \lambda \cdot 1$ that approximates $s$ close enough and is self-adjoint. Let $F$ be the loop completion of the finite subgraph of $E$ consisting of all the edges $e$ of $\alpha$, $\beta$ and all their vertices $r(e)$, $s(e)$ for $\alpha$, $\beta$ appearing in the above expression of $s'$. Let $F_e$ be the exit completion of $F$.

Now we claim that $C^*(F_e)$ is of real rank zero. Considering the ideal $I(\langle S \rangle)$ of $C^*(F_e)$ generated by the sinks $S$ of $F_e$ as in the first paragraph of the proof it suffices to prove that $C^*(G)$ has real rank zero, where $G=(F_e^0\setminus\langle S\rangle, \{e\in F_e^1 \mid r(e)\notin\langle S\rangle\})$ is a subgraph of $F_e$ with no sinks such that $C^*(F_e)/I(\langle S\rangle)\simeq C^*(G)$. Note here that the graph $G$ satisfies condition (K) because $G$ contains all the loops of $F_e$ and the loops of $F_e$ are contained in $F$, which satisfies condition (K) as a loop completion of some subgraph of $E$ with (K). By Theorem 4.2 it is enough to see that $C^*(G)$ has finitely many ideals. We will do this by showing that the number of hereditary subsets of $G^0$ is finite.

The exit completion $F_e$ of $F$ is obtained by adding to $F$ some vertices and some edges. Let $V:=F_e^0\setminus F^0$ be the set of vertices added to $F^0$. Then $V\subset\langle S\rangle$. The set of edges in $F_e$ added to $F$ is the disjoint union of $E^\dagger:=\{e\in F_e^1 \mid r(e)\in V\}$ and $E^\ddagger:=\{e\notin F^1 \mid s(e), r(e)\in F^0$ and $s(e)=s(f)$ for some $f\in F^1\}$. Let $K$ be the subgraph of $F_e$ obtained by removing the vertices in $V$ and all edges in $E^\dagger$, or equivalently that obtained by adding only the edges in $E^\ddagger$ to $F$. The number of hereditary subsets of $K^0$ is finite by Lemma 3.1.

Since $F_e^0=G^0\cup\langle S\rangle$, we have $K^0=F_e^0\setminus V=(G^0\cup\langle S\rangle)\setminus V=G^0\cup(\langle S\rangle\setminus V)$. Now it is easy to show that for a hereditary subset $H$ of $G^0$, $H\cup(\langle S\rangle\setminus V)$ is a hereditary subset of $K^0$ and this correspondence is one to one between the set of hereditary subsets of $G^0$ and the set of hereditary subsets of $K^0$ containing $\langle S\rangle\setminus V$, so that the number of hereditary subsets of $G^0$ is less than or equal to that of $K^0$. Therefore the number of hereditary subsets of $G^0$ is finite.

Now let $B$ be the $C^*$-subalgebra of $C^*(E)$ generated by $\{s_e, p_v : e\in F_e^1, v\in F_e^0\}$. Then the element $s'$ is in $B^\sim$ by the choice of $F_e$. Since $F_e$ is exit complete in $E$, by Proposition 3.1 there exists a $C^*$-homomorphism $h$ from $C^*(F_e)$ onto $B$ such that $h(t_e)=s_e$, $h(q_v)=p_v$, where $\{t_e, q_v\}$ is the Cuntz–Krieger $F_e$-family generating the graph $C^*$-algebra $C^*(F_e)$. Let $s''$ be an element of $C^*(F_e)^\sim$ that can be written in the same form as $s'$ in $C^*(E)^\sim$; $s''=\sum\lambda_{\alpha\beta}t_\alpha t_\beta^*+\lambda\cdot 1\in C^*(F_e)^\sim$. Then there is an invertible self-adjoint element $t$ of $C^*(F_e)^\sim$ that approximates $s''$ close enough since $RR(C^*(F_e)^\sim)=0$. The homomorphism $h$ maps the element $s''\in C^*(F_e)^\sim$ to $s'\in B^\sim$. Thus $s'$ and $h(t)$ are close enough and we conclude that the invertible self-adjoint element $h(t)$ approximates $s$ as close as desired. Therefore $C^*(E)^\sim$ (and so $C^*(E)$) is of real rank zero.

(1) $\Rightarrow$ (3): Note that the property of having real rank zero passes through the quotient algebras and hereditary $C^*$-subalgebras, but clearly $M_n(C(\mathbb{T}))$ does not have real rank zero.

(3) $\Rightarrow$ (2): Suppose $E$ does not satisfy condition (K). Then the subgraph $G$ of $E$ corresponding to the quotient algebra $C^*(E)/I(H)$ clearly does not satisfy condition (K), where $H$ is the hereditary saturation of the sinks of $E$. $G$ has no sinks and there is a simple loop $\alpha=\alpha_1\alpha_2\cdots\alpha_n$

such that $\alpha$ is the only loop based at the vertex $s(\alpha)$. If $\alpha$ has an exit $e$, the range vertex $r(e)$ does not belong to $\alpha^0$. Let $V$ be the hereditary saturation of the vertices $\{r(e) \mid e \text{ is an exit of } \alpha\}$. Then the quotient algebra $C^*(G)/I(V)$ is isomorphic to a graph $C^*$-algebra $C^*(F)$, where $F = (G^0 \setminus V, \{e \in G^1 \mid r(e) \notin V\})$. Note that $F$ contains the loop $\alpha$ and $\alpha$ has no exits in $F$. By [6, Lemma 4.5] $C^*(F)$ contains a projection $p$ such that the corner $pC^*(F)p$ is isomorphic to $M_n(C(\mathbb{T}))$. If $\alpha$ has no exits then $G = F$ and we have the same conclusion. ▮

It is known that a simple, separable, nuclear, unital $C^*$-algebra $A$ is purely infinite if and only if $A \simeq A \otimes \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is the Cuntz algebra generated by isometries $\{S_n \mid n \in \mathbb{N}\}$ with orthogonal ranges (see [8], for example). For nonsimple $C^*$-algebras $A$, three notions of pure infiniteness are introduced in [9] and [10] for the purpose of characterizing nonsimple $C^*$-algebras absorbing the Cuntz algebra $\mathcal{O}_\infty$ (that is, $A \simeq A \otimes \mathcal{O}_\infty$). A $C^*$-algebra $A$ is *purely infinite* if $A$ has no Abelian quotients and every nonzero positive element in $A$ is properly infinite (see [9]). It is proved in [10, Corollary 9.3] that for a separable, nuclear $C^*$-algebras $A$ with an approximate unit consisting of projections $A$ is approximately divisible and purely infinite if and only if $A \simeq A \otimes \mathcal{O}_\infty$. In [6], Hjelmborg proves among others that a graph $C^*$-algebra $C^*(E)$ is purely infinite if and only if every vertex connects to a loop with an exit in every quotient graph $F$. This means that every quotient graph $F$ satisfies condition (K) and every vertex in $F$ connects to a loop in $F$. In particular we have that if $C^*(E)$ is purely infinite then $E$ satisfies condition (K) and hence $C^*(E)$ has real rank zero, but for a separable, nuclear $C^*$-algebra $A$ of real rank zero it is known in [10, Corollary 9.4] that $A$ is purely infinite if and only if $A \simeq A \otimes \mathcal{O}_\infty$. Therefore, with the result [12, Theorem 2.4] that $C^*(E)$ is AF if and only if $E$ has no loops, we have the following corollary.

COROLLARY 4.1. *Let $E$ be a locally finite directed graph with no sinks. Then the following are equivalent.*

1. *$C^*(E)$ is purely infinite,*
2. *$C^*(E) \simeq C^*(E) \otimes \mathcal{O}_\infty$,*
3. *$C^*(E)$ is purely infinite and approximately divisible,*
4. *$C^*(E)$ has no AF quotients and $RR(C^*(E)) = 0$, and*
5. *$C^*(E)$ has no AF quotients and $E$ satisfies condition (K).*

Note also that $C^*(E)$ is purely infinite then every nonzero hereditary $C^*$-subalgebra of $C^*(E)$ contains an infinite projection, which follows from [6, Theorem 3.1] and [12, Theorem 3.9], but this is not true for $C^*$-algebras that are not associated to directed graphs, in general.

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